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COMPARING STOCHASTIC SYSTEMS USING REGENERATIVE SIMULATION WITH--ETC(U)
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COMPARING STOCHASTIC SYSTEMS USING REGENERATIVE SIMULATION WITH COMMON RANDOM NUMBERS*

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1. Introduction

Suppose we have two stochastic systems (perhaps alternative designs for a new system) which are to be compared. Assume that these systems are represented by two regenerative processes $\underset{\sim}{X}(i) = \{X_t(i) : t \geq 0\}$ for $i = 1, 2$; see CRANE and LEMOINE (1977) or IGLEHART (1978) for a discussion of regenerative processes and their role in simulation. Under mild regularity conditions the distribution of $X_t(i)$ converges to the distribution of some limiting random variable (or vector) $X(i)$; this type of convergence is known as weak convergence and written $X_t(i) \Rightarrow X(i)$ as $t \uparrow \infty$. Simulators often speak of $X(i)$ as the steady-state configuration of system i and take as the performance criterion of the system $r_i = E\{f_i(X(i))\}$, where f_i $(i = 1,2)$ are given real-valued functions defined on the state space, $E(i)$, of process $\underset{\sim}{X}(i)$. When comparing the two systems, we wish to estimate the sign of $r_1 - r_2$ by constructing a confidence interval for the quantity.

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Simulation folklore suggests that "common random numbers" be used in this situation in order to reduce variance; see FISHMAN (1973), Section 11.7, and KLEIJNEN (1974) for a discussion of this technique. The basic idea here is to induce a positive correlation between the two systems by simulating the two systems with a common sequence of random numbers. Not only does one save the computer time required to generate the second sequence of random numbers (which would be required if the two systems were simulated independently), but the confidence interval for $r_1 - r_2$ will be shorter provided the positive correlation mentioned above is achieved. Inspite of the generally knowledged appeal of this technique, we know of few published studies which actually carry out the technique and document the savings to be expected; one such study is that of MITCHELL (1973).

When the processes being compared, $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$, are regenerative, we are able to provide a rigorous (asymptotic) analysis of the comparison technique described above. This we do in Section 2 of the paper. While positive correlation is normally expected when using common random numbers, nothing is guaranteed in this respect. One always enjoys the economy of having to generate only half as many random numbers, however, the variance reduction achieved may be minimal or even a variance addition. Conditions for obtaining the desired positive correlation are discussed in Section 3. Section 4 is devoted to several simple stochastic systems which were actually simulated. Here we are able to see exactly what common random numbers buys the simulator in terms of increased computational efficiency. Finally, in Section 5 we state our basic conclusions about the use of common random numbers in comparing stochastic systems using regenerative simulation.

2. Comparing Regenerative Processes

Let $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$ be the two regenerative processes introduced in Section 1. Assume $X_t(i) \Rightarrow X(i)$ as $t \uparrow \infty$ for $i = 1,2$. Given $f_i : E(i) \to R$, we wish to construct a confidence interval for $r_1 - r_2$. The regenerative processes arising most commonly in simulations are discrete and continuous time Markov chains and semi-Markov processes all of which are positive recurrent. An efficient method for reducing the simulation of a continuous time Markov chain (M.C.) to the simulation of a related discrete time M.C. was presented in HORDIJK, IGLEHART, and SCHASSBERGER (1976). The same method can be applied to semi-Markov processes; see IGLEHART (1978). In our simulations presented in Section 4 this method is used. Hence to focus our attention on the comparison problem at hand, we assume that both $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$ are irreducible, aperiodic, positive recurrent Markov chains in discrete time. Under these conditions $X_n(i) \Rightarrow X(i)$ as $n \uparrow \infty$, where $X(i)$ has the stationary (and steady-state) distribution $\underset{\sim}{\pi}(i) = \{\pi_j(i) : j \in E_i\}$. That is, $P\{X(i) = j\} = \pi_j(i)$ for $j \in E(i)$. Then

$$r_i = E\{f_i(X(i))\} = \sum_{j \in E_i} f_i(j)\, \pi_j(i) , \qquad i = 1,2 .$$

Assume for this discussion that $E = E(1) = E(2)$ and that the state $0 \in E$; this is no restriction on our method only a notational convenience. Then let $X_0(i) = 0$, $T_0(i) = 0$, and define the m^{th} entrance to state 0 by $\underset{\sim}{X}(i)$ to be

$$T_m(i) = \inf\{n > T_{m-1}(i) : X_n(i) = 0\} , \qquad m \geq 1 .$$

The times between such entrances are denoted by $\tau_m(i) = T_m(i) - T_{m-1}(i)$, $m \geq 1$, and referred to as the lengths of the 0-cycles for the i^{th} process. Next we compute the area under the function $f_i[X(i)]$ in the m^{th} cycle:

$$Y_m(i) = \sum_{n=T_{m-1}(i)}^{T_m(i)-1} f_i[X_n(i)] , \qquad m \geq 1 .$$

A basic sequence of random variables for the regenerative method is $\{Z_m(i) : m \geq 1\}$, where

$$Z_m(i) = Y_m(i) - r_i \, \tau_m(i) , \qquad m \geq 1 .$$

The regenerative method works because the successive 0-cycles are independent and identically distributed (i.i.d.), which implies that the $Z_m(i)$'s $(m \geq 1)$ are i.i.d., and the following ratio formula holds provided $E\{|f_i(X(i))|\} < \infty$:

$$r_i = E_0\{Y_1(i)\}/E_0\{\tau_1(i)\} .$$

The symbol $E_0\{\cdot\}$ is short for the conditional expectation $E\{\cdot | X_0(i) = 0\}$. For more background on the regenerative method see CRANE and LEMOINE (1977) or IGLEHART (1978). Let $\sigma_i^2 = E_0\{Z_1^2(i)\}$ which we assume is positive and finite. Then two central limit theorems (c.l.t.'s) follow from the regenerative structure of these positive recurrent Markov chains.

One is based on the number of 0-cycles simulated and the other on the number of steps of the M.C. simulated. We define two point estimates of r_i by

$$\hat{r}_i(n) = \left(\frac{1}{n}\sum_{k=1}^{n} Y_k(i)\right)\Big/\left(\frac{1}{n}\sum_{k=1}^{n} \tau_k(i)\right)$$

and

$$\tilde{r}_i(N) = \frac{1}{N}\sum_{k=0}^{N-1} f_i[X_k(i)] \ ,$$

where n is the number of 0-cycles simulated and N is the number of steps of the M.c. simulated. The two c.l.t.'s are the following:

$$n^{1/2}[\hat{r}_i(n) - r_i]/(\sigma_i/E_0\{\tau_1(i)\}) \Rightarrow N(0,1) \tag{2.1}$$

and

$$N^{1/2}[\tilde{r}_i(N) - r_i]/(\sigma_i/E_0^{1/2}\{\tau_1(i)\}) \Rightarrow N(0,1) \tag{2.2}$$

as n and $N \uparrow \infty$. Either (2.1) or (2.2) can be used to construct a confidence interval for r_i.

Now suppose we wish to construct a confidence interval for $r_1 - r_2$ by simulating the two processes $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$ independently; i.e., independent sequences of random numbers will be used to generate the sample paths of the two processes. Form the vectors $\underset{\sim}{r} = (r_1, r_2)$ and $\tilde{\underset{\sim}{r}}(N) = (\tilde{r}_1(N), \tilde{r}_2(N))$. Then we can easily obtain from (2.2) the bivariate c.l.t.

$$N^{1/2}[\tilde{\underset{\sim}{r}}(N) - \underset{\sim}{r}] \Rightarrow N(\underset{\sim}{0}, \underset{\sim}{A}) \ , \tag{2.3}$$

where $N(\underset{\sim}{0}, \underset{\sim}{A})$ is a two-dimensional normal vector with mean vector $\underset{\sim}{0} = (0,0)$ and covariance matrix

$$\underset{\sim}{A} = \begin{pmatrix} \sigma_1^2/E_0\{\tau_1(1)\} & 0 \\ 0 & \sigma_2^2/E_0\{\tau_1(2)\} \end{pmatrix} .$$

An application of the continuous mapping theorem [BILLINGSLEY (1968), Theorem 5.1] to (2.3) yields the following c.l.t. which can be used to construct a confidence interval for $r_1 - r_2$:

$$(2.4) \qquad N^{1/2}[(\tilde{r}_1(N) - \tilde{r}_2(N)) - (r_1 - r_2)]/\sigma \Rightarrow N(0,1) \quad ,$$

where

$$\sigma^2 = \frac{\sigma_1^2}{E_0\{\tau_1(1)\}} + \frac{\sigma_2^2}{E_0\{\tau_1(2)\}} \quad .$$

A c.l.t. comparable to (2.4) but based on 0-cycles can also be obtained.

Our goal in using common random numbers to generate the sample paths of $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$ is to produce a shorter confidence interval for $r_1 - r_2$ for the same length of simulation run (number of steps of the Markov chains generated). In other words we seek a c.l.t. similar to (2.4) but with a smaller value of σ. To accomplish this we generate the bivariate M.c. $\underset{\sim}{X} = \{X_n : n \geq 0\}$, where $X_n = (X_n(1), X_n(2))$. At each jump of the process $\underset{\sim}{X}$ the same random number is used to generate the jumps of the two marginal chains $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$. The marginals of the process $\underset{\sim}{X}$ are seen to have the same finite-dimensional

distributions as the original chains $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$; however, the marginal chains are now dependent. The state space of the chain $\underset{\sim}{X}$ is denoted by F which is a (possibly proper) subset of $E \times E$. We assume here that the chain $\underset{\sim}{X}$ is also irreducible, aperiodic, and positive recurrent. These conditions are not automatic but will usually hold for practical simulations. Furthermore, we assume for convenience that $(0,0) \in F$ and use that state to form regenerative cycles. Note that $X_n \Rightarrow X$ as $n \to \infty$ and the marginal distributions of X are the same as those of $X(1)$ and $X(2)$, namely, $\{\pi_j(i) : j \in E\}$ for $i = 1,2$. For any real-valued function $f : F \to R$ satisfying $E\{|f(X)|\} < \infty$ the regenerative method can be applied to $\underset{\sim}{X}$ to estimate $E\{f(X)\}$. Let $X_0 = (0,0)$ and form $(0,0)$-cycles which begin at the times $T_0 = 0$,

$$T_m = \inf\{n > T_{m-1} : X_n = (0,0)\} , \qquad m \geq 1.$$

Also let $\tau_m = T_m - T_{m-1}$, $m \geq 1$, be the length of the m^{th} cycle and

$$Y_m'(i) = \sum_{n=T_{m-1}}^{T_{m-1}} f_i(X_n) , \qquad m \geq 1 .$$

Here the f-functions are $f(j,k) = f_1(j)$ and $f(j,k) = f_2(k)$. Set $Z_m'(i) = Y_m'(i) - r_i \tau_m$. Since the ratio formula still holds for the process $\underset{\sim}{X}$, $E_{(0,0)}\{Z_m'(i)\} = 0$ for $i = 1,2$. Let

$$\sigma_{ij} = E_{(0,0)}\{Z_1'(i)\, Z_1'(j)\} , \qquad i,j = 1,2$$

which we assume is finite and non-zero. Since the vectors $\underset{\sim}{Z}'_m = (Z'_m(1), Z'_m(2))$ are i.i.d., the standard c.l.t. yields

$$n^{-1/2} \sum_{m=1}^{n} \underset{\sim}{Z}'_m \Rightarrow N(\underset{\sim}{0}, \underset{\sim}{\Sigma}) , \tag{2.5}$$

where $\underset{\sim}{\Sigma} = \{\sigma_{ij}\}$. Just as we are able to go from (2.1) to (2.2) in the one-dimensional case it is possible to obtain from (2.5) the c.l.t.

$$N^{1/2}[\tilde{r}(N) - \underset{\sim}{r}] \Rightarrow N(\underset{\sim}{0},\underset{\sim}{B}) , \tag{2.6}$$

where $\underset{\sim}{B} = \{\sigma_{ij}\, E^{-1}_{(0,0)}\{\tau_1\}\}$. The argument leading to (2.6) is essentially the same as that given by CHUNG (1960), Theorem 16.1. Again using the continuous mapping theorem in conjunction with (2.6) yields

$$N^{1/2}[(\tilde{r}_1(N) - \tilde{r}_2(N)) - (r_1-r_2)]/\nu \Rightarrow N(0,1) , \tag{2.7}$$

where

$$\nu^2 = (\sigma_{11} + \sigma_{22} - 2\sigma_{12})/E_{(0,0)}\{\tau_1\} .$$

A c.l.t. comparable to (2.7) but in terms of n (0,0)-cycles of $\underset{\sim}{X}$ can also be obtained. Now consider the marginals of (2.6) in conjunction with (2.2). Since the marginals of the chain $\underset{\sim}{X}$ have the same stochastic structure as the chains $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$ considered separately, these two c.l.t.'s must be identical. Hence

$$\frac{\sigma_i^2}{E_0\{\tau_1(i)\}} = \frac{\sigma_{11}}{E_{(0,0)}\{\tau_1\}} .$$

Thus upon comparing the constant σ^2 in (2.4) and ν^2 in (2.7) we conclude that $\nu^2 < \sigma^2$ if and only if $\sigma_{12} > 0$. In Section 3 we will examine conditions on the functions f_i and processes $\underset{\sim}{X}(i)$ which guarantee that $\sigma_{12} > 0$.

The measure of variance reduction we use is

$$R^2 = \sigma^2/\nu^2 .$$

So, for example, if $R^2 = 0.5$, then only half as many steps of the Markov chain $\underset{\sim}{X}$ need be simulated to obtain a confidence interval of specified length for $r_1 - r_2$ as would be required when simulating $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$ independently. In addition, of course, only one stream of random numbers need be generated. While we have worked here with discrete time Markov chains, the same method can be used for continuous time Markov chains, semi-Markov processes, and discrete time Markov processes with a general state space. The examples treated in Section 4 illustrate the effectiveness of the method when applied to a variety of these stochastic processes.

3. Guaranteeing Variance Reductions

In this section we investigate conditions under which the variance reduction obtained when $\sigma_{12} > 0$ can be guaranteed. Our major result is that if f_1 and f_2 are monotonic functions (in the same direction) and if $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$ satisfy a stochastic monotonicity condition, then $\sigma_{12} \geq 0$. This result is related to other work on monotonicity and antithetic variates; (see ANDREASSON (1972), MITCHELL (1973), and KLEIJNEN (1974)). When using antithetics (if U is a random variable uniformly distributed on [0,1] then U and 1-U are said to be an antithetic pair) one is generally interested in only one stochastic process, not in comparing the output of two or more processes. Also in the antithetic scheme variance reductions are obtained by creating negative correlation rather than the positive correlation we seek here. If only one process is to be considered, the sample paths of $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$ may be generated using antithetic streams of random numbers. Under proper conditions the results of this section may then be applied to guarantee the desired negative correlation (provided that the two dimensional process $\underset{\sim}{X}$ is regenerative).

The notion of associated random variables can be used to guarantee nonnegative correlation. The following definition and properties may be found in ESARY, PROSCHAN and WALKUP (1967).

(3.1) DEFINITION. Random variables $\underset{\sim}{T} = (T_1, \ldots, T_n)$ are said to be associated if

$$\mathrm{cov}\{f(\underset{\sim}{T}), g(\underset{\sim}{T})\} \geq 0 \quad ,$$

for all nondecreasing functions f and g for which $E\{f(\underset{\sim}{T})\}$, $E\{g(\underset{\sim}{T})\}$, and $E\{f(\underset{\sim}{T})\, g(\underset{\sim}{T})\}$ exist.

(3.2) PROPERTY. Any subset of associated random variables are associated.

(3.3) PROPERTY. If two sets of associated random variables are independent of one another, then their union is a set of associated random variables.

(3.4) PROPERTY. The set consisting of a single random variable is associated.

(3.5) PROPERTY. Nondecreasing functions of associated random variables are associated.

A class of processes for which nonnegative correlation can be guaranteed is stochastically monotone Markov chains (s.m.m.c.). This class was introduced by DALEY (1968) and includes many of the basic queueing models such as the waiting time process in the GI/G/1 queue and the embedded Markov chains used to study the M/G/1 and G/M/s queues. In the following definition let i be a fixed index.

(3.6) DEFINITION. Let $\underset{\sim}{X}(i) = \{X_n(i),\ n \geq 0\}$ be a real valued Markov process with initial distribution $P_i(x) = P\{X_0(i) \leq x\}$ and transition function $P_i(x,A) = P\{X_{n+1}(i) \in A \mid X_n(i) = x\}$ (for measureable sets A). $\underset{\sim}{X}(i)$ is said to be a stochastically Monotone Markov chain if for every y, $P_i(x, (-\infty,y])$ is a nonincreasing function of x.

Define the inverse distribution functions $P_i^{-1}(\cdot)$ and $P_i^{-1}(x,\cdot)$ by

$$P_i^{-1}(u) = \inf\{y : P_i(y) \geq u\} ,$$

$$P_i^{-1}(x,u) = \inf\{y : P_i(x, (-\infty, y]) \geq u\} .$$

Notice that if $\underset{\sim}{X}(i)$ is a s.m.m.c. then $P_i^{-1}(x,u)$ is an increasing function in both arguments. This fact will enable us to show that for each $n \geq 0$ $\{X_0(1), \ldots, X_n(1), X_0(2), \ldots, X_n(2)\}$ are associated.

We shall henceforth assume that the sample paths of $\underset{\sim}{X}(i)$ are generated on the computer using the inverse transformation scheme

$$X_0(i) = P_i^{-1}(U_0) , \tag{3.7}$$

$$X_n(i) = P_i^{-1}(X_{n-1}(i), U_n) , \qquad n \geq 1 \tag{3.8}$$

where $\underset{\sim}{U} = \{U_n, n \geq 0\}$ is a sequence of pseudorandom numbers. $\underset{\sim}{U}$ is, of course, assumed to be a sequence of i.i.d. random variables uniformly distributed on $[0,1]$.

(3.9) THEOREM. If $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$ are both stochastically monotone Markov chains with sample paths generated by (3.7) and (3.8), then for each $n \geq 0$ $\{X_0(1), \ldots, X_n(1), X_0(2), \ldots, X_n(2)\}$ are associated random variables.

PROOF. The proof is by induction. For $n = 0$ (3.4) implies that $\{U_0\}$ is associated and since $P_i^{-1}(U_0)$ is a nondecreasing function of U_0 for each i, (3.5) yields that $\{X_0(1), X_0(2)\}$ are associated. Assume now that $\{X_0(1), \ldots, X_n(1), X_0(2), \ldots, X_n(2)\}$ are associated. Since U_{n+1} is independent of this set, $\{X_0(1), \ldots, X_n(1), X_0(2), \ldots, X_n(2), U_{n+1}\}$ are associated (by (3.3)). The map which takes these random variables into $\{X_0(1), \ldots, X_n(1), X_{n+1}(1), X_0(2), \ldots, X_n(2), X_{n+1}(2)\}$ is nondecreasing because $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$ are both s.m.m.c.'s. Property (3.5) then yields the final result. $\square$

The previous theorem can now be used to show that when simulating s.m.m.c.'s using common random numbers a reduction in variance is obtained.

(3.10) THEOREM. Let $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$ both be stochastically monotone Markov chains with sample paths generated by (3.7) and (3.8). Let f_1 and f_2 be nondecreasing functions. If

(i) $$E\{\tau_m^2\} < \infty ,$$

(ii) $$E\left\{\left(\sum_{n=T_{m-1}}^{T_m-1} |f_i(X_n(i))|\right)^2\right\} < \infty , \qquad \text{for } i = 1, 2,$$

then $\sigma_{12} \geq 0$.

PROOF. Let $S_n(i) = \sum_{K=0}^{n} f_i(X_k(i))$. $S_n(i)$ is then a nondecreasing function of associated random variables so that $\{S_n(1), S_n(2)\}$ are associated. Therefore $\operatorname{cov}\{S_n(1), S_n(2)\} \geq 0$. (This covariance exists and is finite by (i) and (ii), see SMITH (1955).) Theorem 8 of SMITH (1955), which may be applied under assumptions (i) and (ii), implies that

$$\lim_{n \to \infty} \frac{1}{n} \operatorname{cov}\{S_n(1), S_n(2)\} = \sigma_{12} ,$$

so that $\sigma_{12} \geq 0$. □

If f_1 and f_2 are both decreasing then $\sigma_{12} \geq 0$, but if f_1 and f_2 are monotonic in opposite directions, that is if one is increasing and the other is decreasing, then $\sigma_{12} \leq 0$ (in this case antithetics should be used to ensure $\sigma_{12} \geq 0$).

4. Examples

In this section we investigate the magnitude of variance reductions obtained when using common random numbers in simulations of some simple stochastic models. These models are the waiting time process in the M/G/1 queue and the queue length processes in the finite capacity M/M/s queue, repairman problem, and a repairman problem with two repair facilities (sometimes called the central server model). For these models the method was able to produce quite substantial variance reductions. In addition we investigated the use of common random numbers when comparing two different (s,S) inventory policies. To our surprise variance reductions were slight in this case.

The Single Server Queue

Let $W_n(i)$ be the waiting time of the nth customer in the ith GI/G/1 queue which we wish to study. Let $\{S_n(i) : n \geq 0\}$ be the sequence of i.i.d. service times (with mean μ_i^{-1} and distribution function G_i) and $\{A_n(i) : n \geq 1\}$ be the i.i.d. interarrival times (with mean λ_i^{-1} and distribution function F_i) for this queue. Set $X_n(i) = S_{n-1}(i) - A_n(i)$. The waiting times are then defined by

$$W_0(i) = 0$$

$$W_{n+1}(i) = [W_n(i) + X_{n+1}(i)]^+ , \qquad n \geq 0$$

where for any real number a, $[a]^+$ denotes the maximum of 0 and a. Let $\rho_i = \lambda_i/\mu_i$. If $\rho_i < 1$ then $W_n(i) \Rightarrow W(i)$. We shall be interested

in estimating $E\{W(1)\} - E\{W(2)\}$. Recall that $E\{W(i)\}$ is finite if $E\{X_n^+(i)^2\} < \infty$, see KIEFER and WOLFOWITZ (1956). Let $\{V_n : n \geq 0\}$ and $\{U_n : n \geq 1\}$ be independent sequences of i.i.d. uniformly distributed random variables which generate the service and interarrival times by

$$(4.1) \qquad S_n(i) = G_i^{-1}(V_n) \ , \qquad n \geq 0$$

$$(4.2) \qquad A_n(i) = F_i^{-1}(U_n) \ , \qquad n \geq 1 \ ,$$

where G_i^{-1} and F_i^{-1} are the inverse distribution functions of G_i and F_i respectively. The following theorem states conditions under which the two dimensional process $\underset{\sim}{W} = \{(W_n(1), W_n(2)), n \geq 0\}$ will be regenerative.

(4.3) THEOREM. Let $\rho_i < 1$ and $E\{X_n^+(i)^2\} < \infty$ for $i = 1,2$. If the joint distribution of $(X_n(1), X_n(2))$ has a positive density in an open neighborhood of $(0,0)$, then $\underset{\sim}{W}$ is regenerative.

PROOF. Let $\epsilon > 0$ be given and consider the discretized waiting time processes

$$W_0^\epsilon(i) = 0 \ ,$$

$$W_{n+1}^\epsilon(i) = [W_n^\epsilon(i) + X_{n+1}^\epsilon(i)]^+ \ , \qquad n \geq 0$$

where $X_n^\epsilon(i) = k\epsilon$ if $(k-1)\epsilon < X_n(i) \leq k\epsilon$ for some integer k. The process $\underset{\sim}{W}^\epsilon = \{(W_n^\epsilon(1), W_n^\epsilon(2)), n \geq 0\}$ is a Markov chain with a countable

state space. Since $X_n(i) \le X_n^\epsilon(i)$, $W_n(i) \le W_n^\epsilon(i)$ so that returns to (0,0) occur more frequently for $\underset{\sim}{W}$ than for $\underset{\sim}{W}^\epsilon$. The condition on the joint distribution of $X_n(1)$ and $X_n(2)$ ensures that for small enough ϵ, $\underset{\sim}{W}^\epsilon$ will be irreducible and aperiodic (so that $\pi^\epsilon(i,j) = \lim_{n\to\infty} P\{W_n^\epsilon(1) = i\epsilon, W_n^\epsilon(2) = j\epsilon\}$ exists). It therefore suffices to show that $\underset{\sim}{W}^\epsilon$ is positive recurrent.

Let T_m^ϵ be the mth time $\underset{\sim}{W}^\epsilon$ enters (0,0). We seek to show that $E\{T_1^\epsilon\} < \infty$ and since $\pi^\epsilon(0,0) = 1/E\{T_1^\epsilon\}$ we need only show $\pi^\epsilon(0,0) > 0$. Let ϵ be chosen small enough so that the traffic intensity, ρ_i^ϵ, in each discretized queue is less than one and so that $E\{(X_n^\epsilon(i)^+)^2\} < \infty$. Then $W_n^\epsilon(i) \Rightarrow W^\epsilon(i)$ and $E\{W^\epsilon(i)\} < \infty$. Since $\rho_1^\epsilon < 1$,

$$(4.4) \qquad 0 < \pi_1^\epsilon(0) = \lim_{n\to\infty} P\{W_n^\epsilon(1) = 0\}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{\infty} P\{W_n^\epsilon(1) = 0, W_n^\epsilon(2) = k\epsilon\} .$$

Then

$$P\{W_n^\epsilon = (0, k\epsilon)\} \le P\{W_n^\epsilon(2) = k\epsilon\}$$

$$\le P\{W_n^\epsilon(2) \ge k\epsilon\}$$

$$\le P\{W^\epsilon(2) \ge k\epsilon\} ,$$

the last inequality being true because the GI/G/1 queue is a s.m.m.c. But $\sum_{k=0}^{\infty} P\{W^\epsilon(2) \ge k\epsilon\} < \infty$ (since $E\{W^\epsilon(2)\} < \infty$) so that by the dominated convergence theorem we may interchange limits and summation in (4.4). Therefore

$$0 < \sum_{k=0}^{\infty} \pi^{\epsilon}(0,k) \quad ,$$

and since $\pi^{\epsilon}(i,j)$ is either 0 for all i and j or greater than 0 for all i and j it must be true that $\pi^{\epsilon}(0,0) > 0$. □

This theorem may be extended to multiple server queues and to situations in which more than two queues are being considered. In addition these queues possess the proper monotonicity characteristics to ensure that $\sigma_{12} \geq 0$.

The first two sections of Table 1 report variance reductions when two M/G/1 queues are compared. The service times were chosen to have the Weibull distribution:

$$G_i(x) = 1 - \exp\{-(\gamma_i x)^{\alpha_i}\} \quad , \qquad x \geq 0$$

for constants $\gamma_i, \alpha_i > 0$. If $\alpha_i = 1$, G_i reduces to the exponential distribution. The figures in Table 1 are point estimates and 90% confidence intervals based on N independent replications of C cycles of the two dimensional process $\underset{\sim}{W}$. For example, in the section of Table 1 we estimate R^2 to be .142 (and a 90% confidence interval for R^2 is $.142 \pm .015$). Notice that cycles for $\underset{\sim}{W}$ are not much longer than those of the individual processes. In fact for the processes compared in the first section of Table 1 it can be shown that $W_n(1) \leq W_n(2)$ for all n so that $T_m = T_m(2)$. The random number generator described in LEARMONTH and LEWIS (1973) was used for all simulations reported in this paper.

TABLE 1

Point Estimates and 90% Confidence Intervals Using Common Random Numbers. N Replications of C Cycles.

Process Parameters	Process 1	Process 2	r_1	r_2	$E(\tau_1(1))$	$E(\tau_1(2))$	$E(\tau_1)$	R^2	R
model	M/M/1	M/M/1							
λ	.5	.5							
μ	1.5	1.0	.333	.997	1.499	2.005	2.005	.499	.705
ρ	.333	.5	.004	.014	.045	.011	.011	.038	.027
r	.333	1.0							
N = 20,	C = 5000								
model	M/M/1	M/G/1							
λ	.5	.5							
α	1.0	2.0	.996	1.088	2.004	2.450	2.549	.142	.373
γ	1.0	.75	.016	.013	.011	.020	.018	.015	.019
ρ	.5	.59							
r	1.000	1.086							
N = 20,	C = 5000								
model	2 dimensional repairman								
s_1	1	2	8.983	5.351	22.72	57.04	196.0	.447	.658
r	8.994	5.343	.077	.075	.82	1.82	9.2	.063	.045
return state	(9,0)	(4,0)							
N = 20 ,	C = 100								

Continuous Time Markov Chains

The use of common random numbers in comparing two or more continuous time processes is limited by problems in the "synchronization" of the random number streams (see KLEIJNEN (1974). This problem can be overcome in the case of continuous time Markov chains and semi-Markov processes by transforming the continuous time processes into appropriate discrete time Markov chains. Details of this transformation are given in HORDIJK, IGLEHART and SCHASSBERGER (1976). Once in discrete time common random numbers may be used to generate the sample paths of the two processes. This procedure has been used to investigate variance reductions for three finite state space continuous time Markov chains. Because the state spaces are finite the multidimensional processes will always be positive recurrent (assuming irreducibility).

The first two examples, the queue length processes in the finite capacity M/M/s queue and the repairman problem with spares are both birth and death processes. The finite capacity M/M/s queue has birth and death parameters

$$\lambda_i = \begin{cases} \lambda\,, & 0 \le i < M \\ 0\,, & i \ge M \end{cases}$$

$$\mu_i = \begin{cases} i\mu\,, & 1 \le i \le s \\ s\mu\,, & s < i \le M\,, \end{cases}$$

where M is the capacity of the queue. For this model let $\rho = \lambda/s\mu$. The repairman problem has parameters

$$\lambda_i = \begin{cases} n\lambda , & 0 \leq i \leq m \\ (n+m-i)\lambda , & m < i \leq m+n \end{cases}$$

$$\mu_i = \begin{cases} i\mu , & 1 \leq i \leq s \\ s\mu , & s < i \leq m+n \end{cases}$$

where n is the number of operating units, m is the number of spare units, s is the number of repairmen and λ and μ are the failure and repair rates respectively of the units. Calculated variance reductions for these two models are reported in Tables 2 and 3. It should be noted that for our choice of parameters $X_n(1) \leq X_n(2)$ for all $n \geq 0$ provided that $X_0(1) = X_0(2) = 0$. These are examples in which the two dimensional process is not irreducible. In such cases attention must be focused on only one irreducible class of states.

The next example is a multidimensional repairman problem. This example can be modelled by the closed queueing network pictured in Figure 1. A more detailed description of this model may be found in IGLEHART and LEMOINE (1974). The parameters chosen for this model were $n = 10$, $m = 4$, $\lambda = 1$, $p = .2$, $s_2 = 2$, $\mu_2 = 5$, $\mu_1 = 1$. The effect of varying s_1 on the mean number of failed units was studied. Since each individual process has a two dimensional state space, a four dimensional

TABLE 2

Calculated Variance Reductions for Comparing Two Finite Capacity M/M/s Queues, Capacity M = 15

Process Parameters	Process 1	Process 2	r_1	r_2	R^2	R
λ μ s ρ	5 10 1 0.5	5 5 2 0.5	1.00	1.33	.088	.296
λ μ s ρ	5 10 1 0.5	5 3.33 3 0.5	1.00	1.74	.148	.385
λ μ s ρ	5 5 2 0.5	5 3.33 3 0.5	1.33	1.74	.101	.317
λ μ s ρ	9 10 1 0.9	9 3.33 3 0.9	5.11	5.87	.032	.179

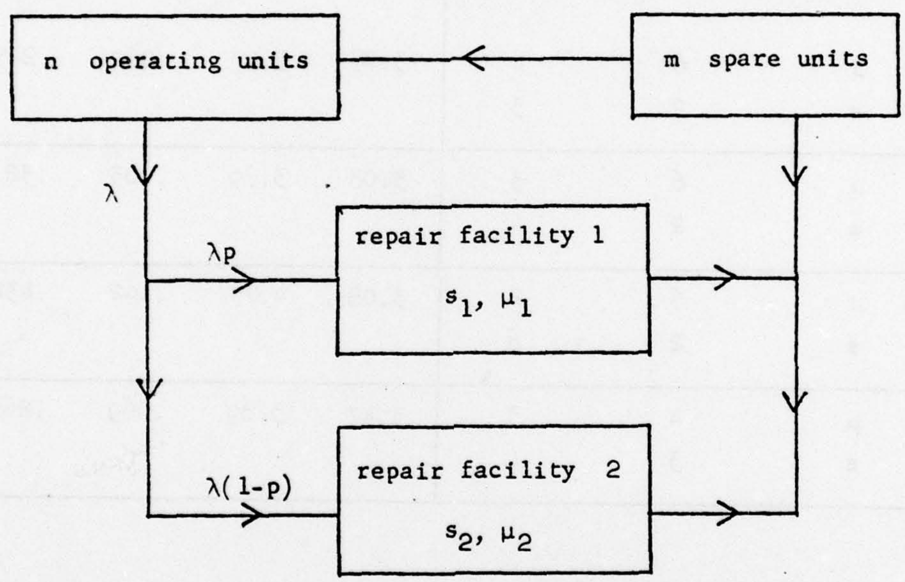
TABLE 3

Calculated Variance Reductions for Comparing Two Repairman Problems, $n = 10$, $m = 4$, $\lambda = 1$.

Process Parameters	Process 1	Process 2	r_1	r_2	R^2	R
μ	6	4	3.08	3.47	.060	.245
s	2	3				
μ	6	3	3.08	3.89	.103	.321
s	2	4				
μ	6	2	3.08	4.75	.192	.432
s	2	6				
μ	4	3	3.47	3.89	.069	.263
s	3	4				

FIGURE 1

Repairman Problem with Two Repair Facilities and Spares



state space is obtained when using common random numbers. The third section of Table 1 reports the estimated variance reduction for this example. The expected cycle length for the four dimensional process is much larger than for either of the two dimensional processes, but it is not as long as we had originally feared. Of course the expected cycle length is a function of the return states chosen and care must be taken so that regenerations do not occur too infrequently.

Inventory Policies

Our final example is comparing two different (s,S) inventory policies. The use of common random numbers should be particularly well suited to inventory problems since it intuitively seems better (i.e., less variable) to compare different policies by subjecting them to the same, rather than independent, demand processes. However, the figures in Table 4 indicated that very little variance reduction is obtained for this model. This apparently is due to the fact that (s,S) policies grossly violate the monotonicity conditions of the previous section.

Let $X_n(i)$ denote the level of stock at the beginning of the nth period and let D_n be the demand during the nth period. Then

$$X_{n+1}(i) = \begin{cases} S, & \text{if } X_n(i) - D_n < s, \\ X_n(i) - D_n, & \text{if } X_n(i) - D_n \leq s. \end{cases}$$

TABLE 4

Calculated Variance Reductions for Comparing Two

(s,S) Inventory Policies, $r_i = E[X(i)]$

Process Parameters	Process 1	Process 2	$E[\tau_1(1)]$	$E[\tau_1(2)]$	$E[\tau_1]$	R^2	R
s	6	5					
S	10	11	3.00	4.00	9.21	.959	.979
r	8.33	8.38					
Demands	Geometric	.5					
s	6	4					
S	10	12	3.00	5.00	11.11	.963	.981
r	8.33	8.40					
Demands	Geometric	.5					
s	5	4					
S	11	12	4.00	5.00	15.21	.967	.984
r	8.38	8.40					
Demands	Geometric	.5					
s	6	5					
S	10	11	2.59	3.46	7.75	.959	.980
r	8.39	8.42					
Demands	Poisson	2					
s	6	4					
S	10	12	2.59	4.32	9.24	.961	.980
r	8.39	8.43					
Demands	Poisson	2					
s	5	4					
S	11	12	3.46	4.32	12.89	.973	.987
r	8.42	8.43					
Demands	Poisson	2					

We choose demands to have either a geometric or Poisson distribution with parameters .5 and 2 respectively.

The discontinuity at s prevents us from obtaining strong positive correlation (and hence good variance reductions). If $X_n(1)$ is near s_1 but $X_n(2)$ is well above s_2 then, with high probability, the first process will increase while the second will decrease. This tends to create negative correlation rather than the desired positive correlation. We were able to increase the variance reductions somewhat by using antithetics when in certain regions of the two dimensional state space. However, the generality of such a procedure seems limited since it may be quite difficult and costly to identify regions of this type which would lead to good variance reductions.

5. Conclusions

In this paper we have shown how the method of common random numbers may be used in certain regenerative simulations to obtain variance reductions. In some cases substantial variance reductions have been obtained, but it seems reasonable to expect that as the complexity of the processes being simulated increases the amount of variance reduction will decrease. It is anticipated that the primary difficulty in the implementation of this method will be the relatively long expected cycle length for the two dimensional process $\underset{\sim}{X}$. Since the validity of the confidence intervals formed will in general depend upon the number of cycles simulated, the method is not suggested for use unless the expected cycle length is short enough so that an adequate number of cycles can be simulated within one's budget constraint. If preliminary simulation runs indicate that the expected cycle length will be excessive (or that the use of the method will result in a variance addition), it is then suggested that independent simulations be performed.

References

[1] ANDREASSON, I.J. (1972). Antithetic methods in queueing simulations. Report NA 72.58, Department of Computer Science, The Royal Institute of Technology, Stockholm, Sweden.

[2] BILLINGSLEY, P. (1968). Convergence of Probability Measures. John Wiley and Sons, New York.

[3] CRANE, M.A. and LEMOINE, A.J. (1977). An Introcution to the Regenerative Method for Simulation Analysis. Springer-Verlag, Berlin.

[4] DALEY, D.J. (1968). Stochastically monotone Markov chains. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete. **10**, 305-317.

[5] ESARY, J.D., PROSCHAN, F. and WALKUP, D.W. (1967). Association of random variables with applications. Ann. Math. Statist. **38**, 1466-1474.

[6] FISHMAN, G.S. (1973). Concepts and Methods in Discrete Event Digital Simulation. John Wiley and Sons, New York.

[7] HORDIJK, A., IGLEHART, D.L. and SCHASSBERGER, R. (1976). Discrete time methods for simulating continuous time Markov chains. Adv. Appl. Prob. **8**, 772-788.

[8] IGLEHART, D.L. and LEMOINE, A.J. (1974). Approximations for the repairman problem with two repair facilities, II: spares. Adv. Appl. Prob. **6**, 147-158.

[9] IGLEHART, D.L. (1978). The regenerative method for simulation analysis. Current Trends in Programming Methodology, III. Software Modeling. K.M. Chandy and R.T. Yeh, eds. Ch. 2, 52-71, Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

[10] KIEFER, J. and WOLFOWITZ, J. (1956). On the characteristics of the general queueing process with applications to random walk. Ann. Math. Statist. **27**, 147-161.

[11] KLEIJNEN, J.P.C. (1974). Statistical Techniques in Simulation, Part I. Marcel Dekker, Inc., New York.

[12] LEARMONTH, G.P. and LEWIS, P.A.W. (1973). Naval Postgraduate School random number generator package LLRANDOM. Naval Postgraduate School Report NP555LW73061A, Monterey, Calif.

[13] MITCHELL, B. (1973). Variance reduction by antithetic variates in GI/G/1 queueing simulations. Operations Res. 21, 988-997.

[14] SMITH, W.L. (1955). Regenerative stochastic processes. Proc. Roy. Soc. London Ser. A. 232, 6-31.

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Suppose two alternative designs for a stochastic system are to be compared. These two systems can be simulated independently or dependently. This paper presents a method for comparing two regenerative stochastic processes in a dependent fashion using common random numbers. A set of sufficient conditions is given that guarantees that the dependent simulations will produce a variance reduction over independent simulations. Numerical examples for a variety of simple stochastic models are included which illustrate the variance reduction achieved.

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